International Journal of Theoretical Physics, Vol. 34, No. 9, 1995

Exact Solutions for Radial Schrödinger Equations

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Received October 14, 1994

An infinite set of exact solutions for the three-dimensional Schrödinger equation with the interactions $ar^2 + br^4 + cr^6$ and $r^2 + \lambda r^2/(1 + gr^2)$ is presented. The conditions under which these solutions can occur are given. Some previously published errors are corrected.

The determination of exact and numerical solutions to the time-independent three-dimensional Schrödinger equation with spherical potentials has been an object of study for many years. The related radial Schrödinger equation for this problem reads

$$\left\{-\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2}V(r) + \frac{l(l+1)}{2r^2}\right\}\psi_l(r) = E\psi_l(r) \tag{1}$$

In the last 15 years many spherical potentials have been considered; two of them find applications in various branches of physics:

(i) The doubly anharmonic oscillator (also called sextic potential),

$$V(r) = ar^2 + br^4 + cr^6 \tag{2}$$

present in studies for structural phase transitions (Khare and Behra, 1980), polaron formation in solids (Amin, 1976, 1982), and the concept of false vacua in field theory (Coleman, 1988).

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0020-7748/95/0900-1931$07.50/0

(ii) The nonpolynomial oscillator

$$V(r) = r^2 + \frac{\lambda r^2}{1 + gr^2} \tag{3}$$

important for its application to field theory in zero dimensions (Salam and Strathdee, 1976) and quantum optics (Haken, 1965).

In a series of papers exact solutions of the Schrödinger equation for the potential (2) have been constructed (Flessas and Das, 1980; Kaushal, 1989; Bose and Varma, 1990; Singh *et al.*, 1990; Parui *et al.*, 1994; Flessas, 1981, 1982). For the potential (3) the eigenvalues for the one-dimensional case have been studied in detail (Flessas, 1981, 1982; Varma, 1981; Lai and Lin, 1982; Whitehead *et al.*, 1982). The three-dimensional case has not been discussed very much. Only Znojil (1991b) considered the lowest lying exact eigenstate in a much broader context. Roy *et al.* (1988) have shown that the nonpolynomial interaction (3) is of a supersymmetric nature if the coupling constants satisfy certain relations. In this way they also obtain the ground-state energy.

In this paper we revisit the above eigenvalue problems in order to correct some of the misunderstandings reported in results for the potential (2) (Singh *et al.*, 1990; Parui *et al.*, 1994) and to extend the results for potential (3). In first instance we follow the technique of Singh *et al.* (1990) to solve (1) for $V(r) = ar^2 + br^4 + cr^6$. We substitute in (1)

$$\psi_l(r) = r^{l+1}\phi(r) \exp\left(-\frac{1}{2}\beta r^2 - \frac{1}{4}\alpha r^4\right) \tag{4}$$

to obtain a second-order equation for $\phi(r)$,

$$\frac{d^2\phi(r)}{dr^2} - 2\left(\beta r + \alpha r^3 - \frac{l+1}{r}\right)\frac{d\phi(r)}{dr} + [\lambda + \sigma r^2]\phi(r) = 0 \tag{5}$$

with

$$\lambda = 2E - \beta(2l + 3) \tag{6}$$

$$\sigma = \beta^2 - a - \alpha(2l + 5) \tag{7}$$

Singh *et al.* (1990) and also Parui *et al.* (1994) proposed for $\phi(r)$ a series of the form

$$\phi(r) = \sum_{k=0}^{\infty} C_k r^k \tag{8}$$

giving rise, after substitution in (5), to the recurrence relation

$$(k + 2)(k + 2l + 3)C_{k+2} + (\lambda - 2\beta k)C_k + (\sigma - 2\alpha(k - 2))C_{k-2} = 0 \tag{9}$$

Since this recurrence relation connects the alternate coefficients C_{k-2}, C_k, and C_{k+2}, the above-cited authors argued that this fact enables them to choose the coefficients C_0 and C_1 independently. First they choose $C_1 = 0$ and $C_0 \neq 0$, by which $\phi(r)$ are only even functions; as a second option they took $C_0 = 0$ and $C_1 \neq 0$, giving rise to odd functions $\phi(r)$. As an example, they tabulate the six lowest lying eigenvalues and the conditions between the coefficients a, b, and c necessary to reach them. This technique is useful if one had checked the existence of a solution of (5) for general l value of the type $\phi(r) = C_0$ and $\phi(r) = C_1 r$. One can easily verify that for $\phi(r) = C_0$ the following conditions have to be fulfilled:

$$\begin{cases} 2E = \beta(2l + 3) \\ -a + \beta^2 = \alpha(2l + 5) \\ b = 2\alpha\beta \\ c = \alpha^2 \end{cases} \tag{10}$$

Out of this one obtains the results already reported in Flessas and Das (1980), Kaushal (1989), Bose and Varma (1990), Singh *et al.* (1990), and Parui *et al.* (1994):

$$E = \frac{\beta}{2}(2l + 3)$$

with

$$b^2 = 4c[a + \sqrt{c}(2l + 5)] \tag{11}$$

We have chosen $\alpha = +\sqrt{c}$ and $\beta = b/(2\sqrt{c})$ in accordance with the discussion by Singh *et al.* (1990). For $\phi(r) = C_1 r$ the following conditions have to be fulfilled:

$$\begin{cases} C_1(l + 1) = 0 \\ C_1(\beta(2l + 5) - 2E) = 0 \\ C_1(-a + \beta^2 - 2\beta(2l + 7)) = 0 \\ C_1(-b + 2\alpha\beta) = 0 \\ C_1(-c + \alpha^2) = 0 \end{cases} \tag{12}$$

The first condition in (11) is essential: either $C_1 = 0$, by which no odd function $\phi(r)$ which is a solution of (5) for all l can be constructed; or $l = -1$ and the problem reduces to the well-known one-dimensional case. Omitting the first condition in (11) results in the second mentioned eigenvalue and solvability condition given in Singh *et al.* (1990) and Parui *et al.* (1994), which are obviously erroneous. This means that no solutions of the type (4) with odd $\phi(r)$ functions for (1) can exist.

In order to complete our results, we also give the eigenvalue and solvability conditions when $\phi(r)$ is of degree two and four, since many of typographic errors are present in Singh *et al.* (1990) and in Parui *et al.* (1994). When $\phi(r) = C_0 + C_2 r^2$ one obtains

$$E = \frac{b}{2\sqrt{c}} \left\{ l + \frac{5}{2} \pm \left[1 + \frac{8c^{3/2}}{b^2}(2l + 3) \right]^{1/2} \right\}$$

with

$$b^2 = 4c(a + \sqrt{c}(2l + 9)) \tag{13}$$

and when $\phi(r) = C_0 + C_2 r^2 + C_4 r^4$

$$E = \frac{b}{2\sqrt{c}} \left[l + \frac{7}{2} + 4\sqrt{q} \cos\left(\frac{1}{3}\theta + 2\pi n \right) \right], \qquad n = 0, 1, 2$$

with

$$q = \frac{1}{3}\left[1 + \frac{16c^{3/2}(l + 2)}{b^2} \right], \qquad \cos(\theta) = \frac{4c^{3/2}}{b^2}\frac{1}{q^{3/2}} \tag{14}$$

The relations (11), (13), and (14) also give the eigenvalue expressions for the one-dimensional case for the even-parity situation ($l = -1$) and the odd-parity one ($l = 0$).

For potential (3) a quite analogous technique can be introduced. In a similar way it is also obvious that exact solutions of (1) of the type

$$\psi_l(r) = r^{l+1}\phi(r)\exp\left(-\frac{r^2}{2} \right) \tag{15}$$

exist and that only even functions $\phi(r)$ can be withheld.

When we choose $\phi(r) = C_0$ the following conditions have to be fulfilled:

$$\begin{cases} C_0(2E - 2l - 3) = 0 \\ C_0(2Eg - 2gl - 3g - \lambda) = 0 \end{cases} \tag{16}$$

This means that either $C_0 = 0$ or $2E = (2l + 3)$ together with $\lambda = 0$. None of these solutions are relevant.

For $\phi(r) = C_0 + C_2 r^2$ three equations occur:

$$\begin{cases} -3C_0 + 2C_0E - 2C_0l + 2C_2(2l + 3) = 0 \\ 2C_2E - 3C_0g + 6C_2g + 2C_0Eg \\ \quad -C_2(2l + 7) - 2C_0gl - \lambda C_0 + 4C_2gl = 0 \\ -7C_2g + 2C_2Eg - 2C_2gl - \lambda C_2 = 0 \end{cases} \tag{17}$$

There are two solutions, of which only one is relevant:

(i) $C_2 = -2C_0/(2l + 3)$ together with $\lambda = 0$, bringing us back to the classical harmonic oscillator problem.

(ii) $C_2 = C_0g$, giving rise to

$$E = (1 - 2g)\left(l + \frac{3}{2}\right)$$

with

$$\lambda = -2g(2 + g(2l + 3)) \tag{18}$$

a result already mentioned in Znojil (1991b) and Roy *et al.* (1988). Notice that up to a normalization factor $\phi(r)$ is proportional to $(1 + gr^2)$.

With $\phi(r) = C_0 + C_2r^2 + C_4r^4$ the computation becomes quite complex. A set of four equations has to be solved:

$$\begin{cases} 2Eg - \lambda - 11g - 2gl = 0 \\ 2C_4E - \lambda C_2 - 7C_2g + 20C_2g + 2C_2Eg \\ \quad - C_4(2l + 11) - 2C_2gl + 8C_4gl = 0 \\ 2C_2E - \lambda C_0 + 20C_4 - 3C_0g + 6C_2g + 2C_0Eg - C_2(2l + 7) \\ \quad + 8C_4l - 2C_0gl + 4C_2gl = 0 \\ 2C_0E + 6C_2 - C_0(2l + 3) + 4C_2l = 0 \end{cases} \tag{19}$$

Solving these equations delivers three solutions for λ:

$$\lambda = 0$$

relating again the potential to the harmonic oscillator one, and

$$\begin{aligned} \lambda = \tfrac{1}{2}g\{&-(12 + 26g + 12gl) \\ &\pm [(12 + 26g + 12gl)^2 \\ &- 4(32 + 160g + 120g^2 + 64gl + 128g^2l + 32g^2l^2)]^{1/2}\} \end{aligned}$$

giving rise to two eigenvalues

$$E = \frac{5}{2} - \frac{13g}{2} + l - 3gl \pm \frac{(4 - 4g + 49g^2 + 8gl + 28g^2l + 4g^2l^2)^{1/2}}{2} \quad (20)$$

Notice again that the one-dimensional eigenvalues (Flessas, 1981, 1982; Varma, 1981; Lai and Lin, 1982; Whitehead *et al.*, 1982) follow directly from the above expressions when replacing l either by -1 or 0.

The potentials (2) and (3) belong to more general classes of three-dimensional anharmonic potentials, for which eigenvalues in a closed form can be derived. It is mentioned by Flessas and Das (1980) that the potential

$$V(r) = \omega^2 r^2 + a_2 r^4 + a_3 r^6 + \cdots + a_{2n+1} r^{2(2n+1)} \quad (21)$$

can have eigenfunctions of the general type

$$\psi(r)_l(r) = r^{l+1}\phi(r) \exp(-b_1 r^2 - b_2 r^4 - \cdots - b_{n+1} r^{2(n+1)}) \quad (22)$$

provided n relations hold among the $2n$ constants $a_2, a_3, \ldots, a_{2n+1}$. Only the case for $n = 2$ is worked out to a certain extent. Similarly Znojil (1991a) and Hislop *et al.* (1990) discussed potentials

$$V(r) = \omega r^2 + \frac{f(r)}{g(r)} \quad (23)$$

where $f(r)$ and $g(r)$ are polynomials in r^2 of degree N, also called Padé anharmonic oscillators. Only a few very simple examples are touched upon by Znojil (1991a) and Hislop *et al.* (1990). In order to obtain closed expressions for eigenvalues many relations between the coefficients of the several powers in r in $f(r)$ and $g(r)$ have to be fulfilled.

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